Written Exam for the M.Sc. in Economics Autumn 2015 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: **February 12, 2016**

3-hour closed book exam.

Please note there are a total of **9** questions which should **all be replied to**. That is, **4** questions under *Question A*, and **5** under *Question B*.

Total numbers of pages (including this one): 5

Please also note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish. If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Question A:

Consider the ARCH model given by,

$$x_t = \sigma_t \eta_t, \qquad t = 1, 2, ..., T \tag{1}$$

with η_t i.i.d.N$(0,1)$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta z_{t-1}^2. \tag{2}$$

Here z_t is some exogenous covariate, as for example the realized volatility.

Question A.1: Suppose that $\beta = 0$ and recall that $E\eta_t^4 = 3$. Derive a condition for x_t to be weakly mixing and such that $Ex_t^4 < \infty$.

Question A.2: Now consider the case of $\beta > 0, \omega > 0$ and $\alpha \geq 0$. Assume that also z_t is i.i.d.N$(0, \sigma_z^2)$, and that z_t and η_t are independent. Define the bivariate vector $v_t = (x_t, z_t)'$ and observe that the density of v_t conditional on v_{t-1} is given by,

$$f(v_t | v_{t-1}) = \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_z^2 \sigma_t^2}} \exp\left(-\frac{1}{2}\left\{\frac{x_t^2}{\sigma_t^2} + \frac{z_t^2}{\sigma_z^2}\right\}\right). \tag{3}$$

Argue that v_t is a Markov chain for which the transition density $f(\cdot|\cdot)$ is such that the drift criterion can be applied.

Next, with drift function $\delta(v_t) = 1 + v_t' v_t = 1 + x_t^2 + z_t^2$ and $v = (x, z)'$, show that for some constant c

$$E(\delta(v_t) | v_{t-1} = v) \leq c + \max(\alpha, \beta)(x^2 + z^2). \tag{4}$$

Conclude that if $\max(\alpha, \beta) < 1$, then v_t is weakly mixing with $E\|v_t\|^2 \leq E[x_t^2] + E[z_t^2] < \infty$.

Question A.3: With $L_T(\omega, \alpha, \beta)$ the log-likelihood function for the ARCH model, it holds that the score for β is given by,

$$S(\omega, \alpha, \beta) = \partial \log L_T(\omega, \alpha, \beta) / \partial \beta = \sum_{t=1}^{T} \frac{1}{2}\left(\frac{x_t^2}{\sigma_t^2} - 1\right) \frac{z_{t-1}^2}{\sigma_t^2}. \tag{5}$$

Show that with $\omega_0 > 0$, $\alpha_0 < 1$ and $0 < \beta_L \leq \beta_0 < 1$ then under the condition that $v_t = (x_t, z_t)'$ is weakly mixing,

$$\frac{1}{\sqrt{T}} S(\omega_0, \alpha_0, \beta_0) \xrightarrow{d} N\left(0, \frac{\nu}{2}\right), \tag{6}$$

where $\nu = E[(z_{t-1}^2 / (\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 z_{t-1}^2))^2] \leq 1/\beta_L^2 < \infty$.

Question A.4: With z_t Realized volatility for S&P500 and x_t log-returns on S&P500, ML estimation gave:

Output: MLE of ARCH with RV	
$\hat{\alpha} = 0.11$	std.deviation($\hat{\alpha}$) $= 0.012$
$\hat{\beta} = 0.09$	std.deviation($\hat{\beta}$) $= 0.091$

What would you conclude in terms of the importance of Realized volatility?

Question B:

Consider the switching-ARCH(1) model given by

$$\begin{aligned} y_t &= \sigma_t z_t \\ \sigma_t^2 &= \omega_0 + \omega_1 1_{(S_t=1)} + \alpha y_{t-1}^2 \end{aligned}$$

where z_t and S_t are independent, with z_t i.i.d.N$(0,1)$ and S_t can take value 1 or 2. Note that $1_{(S_t=1)} = 1$ if $S_t = 1$ and $1_{(S_t=1)} = 0$ if $S_t = 2$. Moreover, $\omega_0 > 0$, $\omega_1 \geq 0$, and $\alpha \geq 0$.

Question B.1: Suppose that $\alpha = \omega_1 = 0$. Explain if y_t is weakly mixing.

Question B.2: Next, assume that S_t is a Markov chain evolving according to the transition probabilities $p_{ij} = P(S_t = j | S_{t-1} = i)$, $i, j = 1, 2$ where the transition probabilities p_{ij} are such that S_t is weakly mixing.

Suppose that $\alpha = 0$ while $\omega_1 > 0$. Explain if σ_t^2 is weakly mixing. Is y_t weakly mixing?

Question B.3: Suppose that S_t is i.i.d. with $P(S_t = 1) = p$ and $P(S_t = 2) = 1 - p$. State the density of y_t given y_{t-1} and $S_t = 1$. That is, find

$$f(y_t | y_{t-1}, S_t = 1). \tag{7}$$

Likewise, find $f(y_t | y_{t-1}, S_t = 2)$.

Question B.4: We want to estimate the model parameters $\theta = (\omega_0, \omega_1, \alpha, p)$ based on the EM algorithm. First, we seek to compute the EM log-likelihood function $L_{EM}(\theta)$ which we use in the expectation step (the E-step).
Treating $(S_t)_{t=1}^T$ as observed variables, consider the infeasible log-likelihood function defined as,

$$\begin{aligned} L(y_1, ..., y_T, S_1, ..., S_T; \theta) = \sum_{t=2}^{T} \big\{ & 1_{(S_t=1)} [\log f(y_t | y_{t-1}, S_t = 1) + \log(p)] \\ & + 1_{(S_t=2)} [\log f(y_t | y_{t-1}, S_t = 2) + \log(1-p)] \big\} . \end{aligned}$$

Recall that the E-step relies on making a guess of θ, $\theta = \tilde{\theta}$ say, and next computing

$$L_{EM}(\theta) = E_{\tilde{\theta}}[L(y_1, ..., y_T, S_1, ..., S_T; \theta) | y_1, ..., y_T].$$

This includes the computation of

$$P_t^*(1) := E_{\tilde{\theta}}[1_{(S_t=1)}|y_1, ..., y_T] = f_{\tilde{\theta}}(S_t = 1|y_1, ..., y_T),$$

where $f_{\tilde{\theta}}(S_t = 1|y_1, ..., y_T)$ denotes the probability (or density) $f(S_t = 1|y_1, ..., y_T)$ evaluated at $\tilde{\theta}$.

Show that, under the conditions in Question B.3 that for the case of $t = 2$,

$$f(S_2 = 1|y_1, y_2, ..., y_T) = \frac{f(y_2, ..., y_T|S_2 = 1, y_1)f(S_2 = 1, y_1)}{\sum_{i=1}^{2} f(S_2 = i, y_1, ..., y_T)}.$$

Question B.5: Using the above, and with $P_t^*(2) = f_{\tilde{\theta}}(S_t = 2|y_1, ..., y_T)$ it follows that

$$L_{EM}(\theta) = \sum_{t=2}^{T} \{P_t^*(1)[\log f(y_t|y_{t-1}, S_t = 1) + \log(p)] \\ + P_t^*(2)[\log f(y_t|y_{t-1}, S_t = 2) + \log(1-p)]\}.$$

Explain how this EM-log-likelihood function can be used to find an estimate of θ.